To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism

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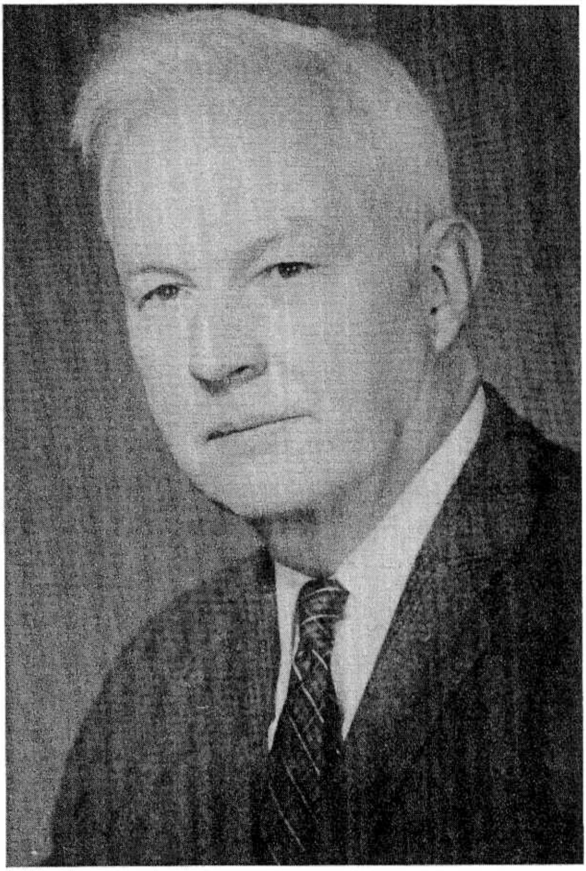

Dedicated to Haskell B. Curry on the occasion
of his 80th Birthday

COMBINATORY AND LAMBDA SYNTAX

OPTIMAL REDUCTIONS IN THE LAMBDA-CALCULUS

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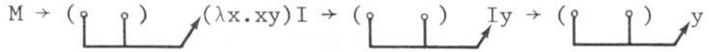
1. INTRODUCTION

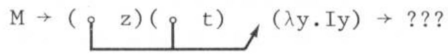
The standardisation theorem [Cu] implies that leftmost-outermost reductions reach normal forms whenever they exist. Thus these reductions insure termination and can be called correct in the sense of [Vu,Do,Le1]. However, they may not be optimal, i.e. reach the normal form in a minimum number of steps because of duplications of redexes. At first glance, it seems difficult to design optimal reduction strategies. First, innermost redexes seem better than the outermost ones because they avoid copying redexes. Secondly, one does not want to reduce useless redexes, i.e. redexes not reduced in the leftmost-outermost reduction, but they cannot be found effectively in the λ-calculus. Worse, even in the λI-calculus where all redexes are useful for the normal form, innermost reductions may not be optimal. Take for instance $(\lambda x.xI)(\lambda y.(\lambda z.zzzz)(yt))$ with $I=\lambda x.x$. However, some optimal strategies have been defined in [Vu] for recursive programs schemes. Independently, an efficient, though not optimal, strategy was designed by [Wa] for the λ-calculus. Both methods use the same simple principle : at each reduction step, contract the leftmost-outermost redex and avoid duplications of redexes by adding some sharing mechanism. This is done by leaving the usual universe of λ-expressions and by considering shared λ-expressions. For instance, let M=Δ((λx.xy)I) with Δ=λx.xx and I=λx.x. The graph normal reduction method in [Wa] works for M as follows :

(i) 

Thus both copies of (λx.xy)I and Iy are simultaneously contracted. The corresponding reduction in the usual λ-calculus framework is :

(ii) M → ((λx.xy)I)((λx.xy)I) → (Iy)(Iy) → yy.

But shared reductions are not so easy in the λ-calculus because of functional arguments (which cannot happen in recursive program schemes considered in [Vu]). Take for instance M=(λx.(xz)(xt))(λy.Iy). Then :

(iii) 

The leftmost outermost redex (λy.Iy)z has to be reduced without duplicating the Iy redex. The evaluator of [Wa] cannot handle this example and makes copies of (λy.Iy) in this case, which makes it non-optimal. Example (iii) shows that, in order to get an optimal λ-evaluator, one must share, not only subexpressions, but pairs of subexpressions and substitutions for free variables, i.e. <u>closures</u> in the programming languages terminology. To our knowledge, such optimal evaluators have not yet been designed for the λ-calculus, and Wadsworth's method is still the most efficient one. Unfortunately, we shall not give here a solution to that problem. But we will characterize exactly the redexes whose contraction needs to be shared. For doing this, we shall study the duplication of redexes in the usual λ-calculus setting.

Take again example (i). In the corresponding reduction (ii), at each step the leftmost-outermost redex is at least contracted. But in the second and third steps, some additional redexes are reduced. In the second step, the two ((λx.xy)I) contracted redexes are copies (or residuals) of one single redex in M. In step three, the two Iy redexes are not residuals of one redex in reduction (ii), but they are residuals of a single redex if one permutes the two first steps, which gives reduction

(iv) $M \to \Delta(Iy) \to (Iy)(Iy) \to yy$.

This key observation leads to the organisation of our paper. First, we recall definitions and properties of permutations of reductions. The full treatment of them has been done in [Be1] for recursive programs schemes, but is quite similar in the λ-calculus. Secondly, we formalize duplications of redexes as residuals modulo permutations of reductions. We show that its symmetric and transitive closure, called the family relation, is decidable. This implies that one can effectively find maximum sets of redexes which are duplications of a single redex. Thus complete reductions, i.e. reductions which contract at each step such maximum sets, are effective reduction strategies and can be proved finally optimal if they contract, at each step, at least leftmost-outermost redexes.

Readers interested in a more complete treatment are sent to [Le2]. A similar paper for the easier formalism of recursive programs schemes is [Be1], where results of [Vu] are shown along the lines of this paper. Also, optimal reductions have been studied in [O'D] (although we disagree with some of his results [Be2]) and in [St] for combinatory logic (which is an easier case than the λ-calculus).

2. PRELIMINARIES

If V is an infinite set of variables, the set of λ-expressions built on V is the minimum set containing V and closed by abstraction and application, i.e. (λx.M) and (MN) are λ-expressions when M,N are already λ-expressions and x is a variable. Parentheses are suppressed as much as possible. Around applications, parentheses may be omitted by association to the left. There is the usual notion of free and bound variables. We do not take care of names of bound variables and thus forget all the meaningless problems of the so-called α-rule. Therefore, equality of λ-expressions will be equality modulo α-interconvertibility and, for instance, we do not hesitate to write $\lambda x.xy=\lambda t.ty$.

Here, only the β-rule is considered. Let say that M can be immediately reduced to N, written $M \to N$, iff M and N only differ by a subexpression which is of the form $(\lambda x.P)Q$ in M and $P[x\backslash Q]$ in N (where $P[x\backslash Q]$ is the result of substituting Q to all occurrences of the free variable x in P). A subexpression of the form $(\lambda x.P)Q$ is called a redex. The transitive closure of $\to$ is written $\overset{*}{\to}$ and, thus, $M \overset{*}{\to} N$ means that M can be reduced to N. An expression without redexes is a normal form and, if $M \overset{*}{\to} N$ with N in normal form, then M has a normal form.

Several reductions are possible between two expressions. If one wants to be more specific about one reduction, names can be given to them. We use letters ρ, σ, τ. And one has to specify the initial expression and the successive redex-occurrences contracted at each step. In order to be very precise, one needs some addressing mechanism of redexes in λ-expressions. Here we avoid it by confusing redexes and their occurrences and by always assuming that this distinction is clear from the context. Thus we write $\rho : M \xrightarrow{R} N$ to say that ρ is the immediate reduction consisting in contracting the redex (occurrence) R in M. More generally, a reduction ρ can be specified by writing :

(v) $\rho : M \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \dots \xrightarrow{R_n} M_n$.

We use also the notation $\rho : M \overset{*}{\to} N$ for meaning that ρ is a reduction from M to N. By $\rho\sigma$, we mean the obvious composition of ρ and σ. The empty reduction starting at M will be written 0_M or simply 0, when M is clear from the context. Similarly, we can forget the initial expression of an immediate reduction and write just the redex (occurrence) R instead of $M \xrightarrow{R} N$. Thus (v) is equivalent to $\rho: M \overset{*}{\to} N$ and $\rho = R_1 R_2 \ldots R_n$. Letters R,S,T will be reserved to redex-occurrences.

Now, suppose that R is a redex in M and $\rho: M \overset{*}{\to} N$. Then R can be copied, modified, eliminated or contracted during ρ. The set R/ρ of redexes corresponding to R is called the set of residuals of R by ρ. First, one has :

$$R/0 = \{R\}$$

$$R/\rho\sigma = \{T \mid T \in S/\sigma, S \in R/\rho\}$$

If $\rho: M \xrightarrow{S} N$, then $R/\rho = R/S$ is defined by one tedious inspection of the relative positions of R and S. If R is not contained in S, then $R/S = \{R'\}$ where R' is the redex of N which is at the same place as R in M. If R coincides with S, then $R/S = \emptyset$. If R is strictly in $S = (\lambda x.A)B$, there are two possibilities. First R is in A, and $R/S = \{R'\}$ is the redex which corresponds to R in the contractum $A[x\backslash B]$ of S in N. (Remark then that $R' = R[x\backslash B]$). Otherwise R is in B and $R/S = \{R_1, R_2, \ldots R_n\}$ where n is the number of occurrences of the free variable x in A and every R_i corresponds to R in the i^{th} instance of B in the contractum $A[x\backslash B]$ of S in N.

Finally, it will be necessary to consider another kind of reductions, parallel reductions. Let F be a set of (maybe nested) redexes in M. The immediate parallel reduction $M \xrightarrow{F} N$ can be defined by use of the following definitions and theorem. A reduction $\rho = R_1 R_2 \ldots R_n \ldots$ is relative to F iff $\forall n \geq 1 \;\; R_n \in F/(R_1 R_2 \ldots R_{n-1})$. Moreover ρ is a development of F iff ρ is relative to F and $F/\rho = \emptyset$. Then

THEOREM 2.1. (Finite developments theorem) : [Cu]. Let F be a set of redexes in an expression M. Then

1) there is no infinite reduction relative to F,

2) all developments end at a same expression,

3) for all redex R in M, if ρ and σ are two developments of F, then $R/\rho = R/\sigma$.

Thus, the order in which redexes of F are contracted is not relevant and parallel reductions can be defined now without ambiguity as reductions $M \xrightarrow{F_1} M_1 \xrightarrow{F_2} M_2 \ldots \xrightarrow{F_n} M_n$ contracting some set of redexes at each step. Non parallel reductions are a particular case of parallel ones since, at each step, a singleton set of redexes is contracted. We generalize all the previous notations to parallel reductions without difficulty.[1] In the rest of the paper, only parallel reductions are considered. Therefore we simply call them reductions. But we do not hesitate to write $\rho = R_1 R_2 \ldots R_n$ for $\rho = \{R_1\}\{R_2\}\ldots\{R_n\}$. Remark finally that $\emptyset \neq 0$, i.e. one immediate contraction of an empty set of redexes is not an empty (parallel) reduction.

3. PERMUTATIONS OF REDUCTIONS

If F and C are two sets of redexes in M, let $F \sqcup C = F(C/F)$. Then one useful corollary of the finite developments theorem is

LEMMA 3.1. (Lemma of parallel moves) [Cu] : Let F and C be two sets of redexes in M. Then

1) $F \sqcup C$ and $C \sqcup F$ end at the same expression,
2) $H/(F \sqcup C) = H/(C \sqcup F)$ for any set H of redexes in M.

This can be summarized in figure 1. Now, permutations of reductions may be defined.

Definition 3.2. The equivalence of reductions by permutations is the least congruence with respect to composition satisfying the lemma of parallel moves and elimination of empty steps.

More explicitly, this relation $\equiv$ is the least equivalence relation satisfying :

1) $F \sqcup C \equiv C \sqcup F$ when F and C are two sets of redexes in a same expression,

2) $\emptyset \equiv 0$,

3) $\rho\sigma\tau \equiv \rho\sigma'\tau$ if $\sigma \equiv \sigma'$.

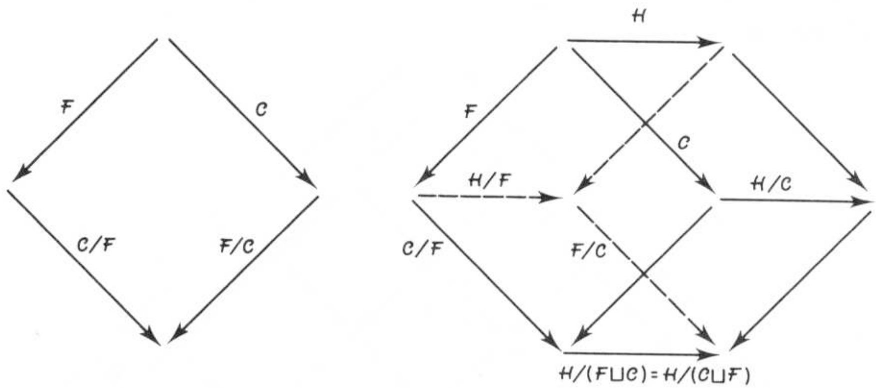


Figure 1

Similarly, an embedding relation can be defined by stating : $\rho \underline{[} \sigma$ iff $\exists\tau.\ \rho\tau\equiv\sigma$. Fortunately, both embedding and permutation equivalence relations can be proved effective by extending the residual definition to reductions. (Remark that $\rho\equiv\sigma$ is not clearly decidable since the length of ρ and σ may differ because of the elimination of empty steps).

Suppose ρ and σ are two reductions which start at a same expression. Then the reduction residual σ/ρ of σ by ρ is a reduction starting at the end of ρ which is defined inductively on the sum of length of ρ and σ by :

$$0/\rho=0$$

$$(\sigma F)/\rho=(\sigma/\rho)(F/(\rho/\sigma))$$

From this definition, some easy algebraic properties of residuals of reduction can be shown. For instance :

$$(\sigma\tau)/\rho=(\sigma/\rho)(\tau/(\rho/\sigma))$$
$$\rho/(\sigma\tau)=(\rho/\sigma)/\tau$$
$$\rho/0=\rho$$

A simpler way of considering residuals of reductions is to iteratively apply the square diagram of figure 1. Thus one gets figure 2.

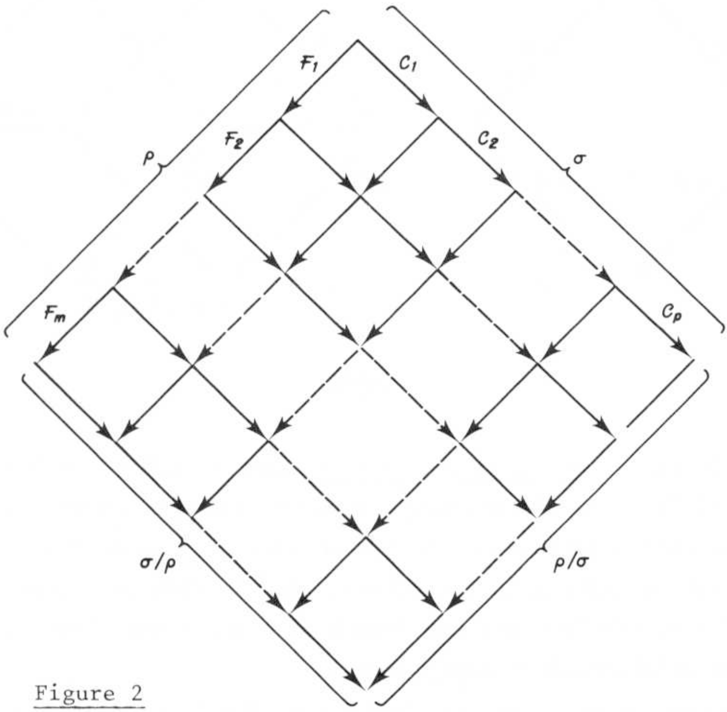


Figure 2

Now, we extend obviously the parallel moves lemma. If ρ and σ start at the same expression, let $\rho\sqcup\sigma=\rho(\sigma/\rho)$. Then :

LEMMA 3.2. Let ρ and σ be two reductions starting at M. Then

1) $\rho\sqcup\sigma$ and $\sigma\sqcup\rho$ end at the same expression,
2) $\tau/(\rho\sqcup\sigma)=\tau/(\sigma\sqcup\rho)$ for any other reduction τ starting also at M.

Proof : obvious iteration of lemma 3.1. □

Let $\emptyset_M^n$ be the reduction consisting of n empty steps starting at M, and $\emptyset^n=\emptyset\emptyset\ldots\emptyset$ (n times) when M is assumed from the context. Furthermore, let $|\rho|$ be the length of ρ (i.e. its number of steps). Now, one can state the easy decision procedures for relations $\equiv$ and $\sqsubseteq$.

LEMMA 3.3. Let ρ and σ be two reductions starting at M. Then

1) $\rho\equiv\sigma$ iff $\rho/\sigma=\emptyset^m$ and $\sigma/\rho=\emptyset^n$ with $m=|\rho|, n=|\sigma|$,

2) $\rho\sqsubseteq\sigma$ iff $\rho/\sigma=\emptyset^m$ with $m=|\rho|$.

Proof : by using properties of residuals of reductions and the previous lemma. Remark first that, when ρ,σ,τ are coinitial, one has $(\rho\sqcup\sigma)/\tau=(\rho/\tau)\sqcup(\sigma/\tau)$. Thus one can prove by induction on the definition of $\equiv$ that $\rho\equiv\sigma$ implies $\tau/\rho=\tau/\sigma$ for all τ. Thus $\rho\equiv\sigma$ implies $\rho/\sigma=\rho/\rho$ and $\sigma/\rho=\sigma/\sigma$. Therefore $\rho/\sigma=\emptyset^m$, $\sigma/\rho=\emptyset^n$ where $m=|\rho|$, $n=|\sigma|$.

Now, if $\rho\sqsubseteq\sigma$, there is τ such that $\rho\tau\equiv\sigma$. Hence $\rho\tau/\sigma\equiv\emptyset^p$ with $p=|\rho\tau|$, which implies $\rho/\sigma=\emptyset^n$ for some n. As $|\rho/\sigma|=|\rho|$, one has $n=|\rho|$.

Conversely, suppose $\rho/\sigma=\emptyset^m$ and $\sigma/\rho=\emptyset^n$ for some m,n. Then $\rho\equiv\rho\sqcup\sigma$ and $\sigma\equiv\sigma\sqcup\rho$ by elimination of empty steps. But $\rho\sqcup\sigma\equiv\sigma\sqcup\rho$ for any coinitial ρ and σ. And $\rho\equiv\sigma$. Similarly, when $\rho/\sigma=\emptyset^m$, let $\tau=\sigma/\rho$. Then $\rho\tau=\rho\sqcup\sigma\equiv\sigma\sqcup\rho\equiv\sigma$. □

Two equivalent reductions are reductions with the same initial and final expressions, but the converse may not be true. Take for instance M=I(Ix) with I=λx.x . There are two redexes R and S in M, but reductions $M\xrightarrow{R}Ix$ and $M\xrightarrow{S}Ix$ are not equivalent since $R/S\neq\emptyset$ and $S/R\neq\emptyset$. Similarly, if $\Delta=\lambda x.xx$ and R is the only redex in $\Delta\Delta$, one gets $0\not\equiv R\not\equiv RR\not\equiv RRR\ldots$ (However $0\sqsubseteq R\sqsubseteq RR\sqsubseteq RRR\ldots$). In figure 3, some more complicated example is treated.

Some easy properties of $\equiv$ and $\sqsubseteq$ can be proved by algebraic manipulations. We give a list of them (forgetting the appropriate conditions on initial and final expressions of reductions).

$\rho \sqcup \sigma \equiv \sigma \sqcup \rho$,

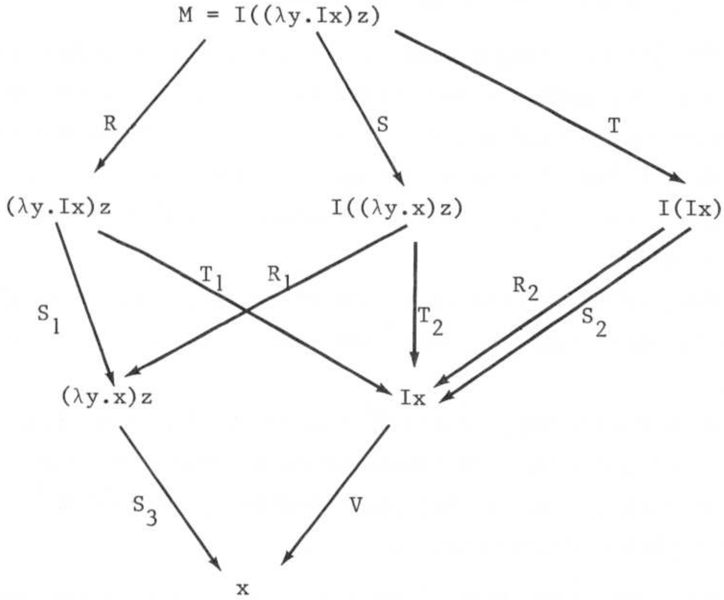
$\rho \equiv \sigma$ iff $\forall \tau .\ \tau/\rho = \tau/\sigma$,

$\rho\sigma \equiv \rho\tau$ iff $\sigma \equiv \tau$,

$\rho \equiv \sigma$ implies $\rho/\tau \equiv \sigma/\tau$,

$\rho \sqsubseteq \rho$,

$\rho \sqsubseteq \sigma \sqsubseteq \tau$ implies $\rho \sqsubseteq \tau$,



$$R_2 \not\equiv S_2$$

$$R \sqcup S = RS_1 \equiv SR_1 = S \sqcup R$$

$$R \sqcup T = RT_1 \equiv TR_2 = T \sqcup R$$

$$S \sqcup T = ST_2 \equiv TS_2 = T \sqcup S$$

$$RT_1 \not\equiv ST_2$$

<u>Figure 3</u>

$\rho \sqsubseteq \sigma \sqsubseteq \rho$ iff $\rho \equiv \sigma$,

$\rho \sqsubseteq \sigma$ implies $\rho/\tau \sqsubseteq \sigma/\tau$,

$\rho\sigma \sqsubseteq \rho\tau$ iff $\sigma \sqsubseteq \tau$,

$\rho \sqsubseteq \rho \sqcup \sigma$,

$\sigma \sqsubseteq \rho \sqcup \sigma$,

$\rho \sqsubseteq \tau$ and $\sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \equiv \sigma \sqcup \rho \sqsubseteq \tau$.

Thus, $\sqsubseteq$ is a preorder with $\equiv$ as associated equivalence. Also $\equiv$ is a congruence for / and hence for $\sqcup$. Moreover $\sqsubseteq$ and $\equiv$ are left-simplifiable. Finally, $\sqsubseteq$ induces some semi sup-lattice structure on coinitial reductions quotiented by $\equiv$. Thus, we find in a very elementary way the computation lattice exhibited in recursive programs schemes by [Vu]. However, his proof was more complicated and holds only with certain restrictions, which permit to identify $\rho \sqsubseteq \sigma$ with $N \overset{*}{\to} P$ whenever $\rho : M \overset{*}{\to} N$ and $\sigma : M \overset{*}{\to} P$. All our study can be rephrased in category theory terminology [Pℓ]. We have two remarks.

First, it is not true that $\sqsubseteq$ induces some lattice property in the λ-calculus. Two coinitial reductions may not have some greatest lower bound. Take for instance $M=(\lambda x.K_a(xY))K_b$ with $K_a=\lambda x.a$, $K_b=\lambda x.b$, $Y=(\lambda x.f(xx))(\lambda x.f(xx))$. Then

$$\rho : M \to K_a(K_bY) \to K_a b$$

$$\sigma : M \to (\lambda x.a)K_b$$

have no glb.

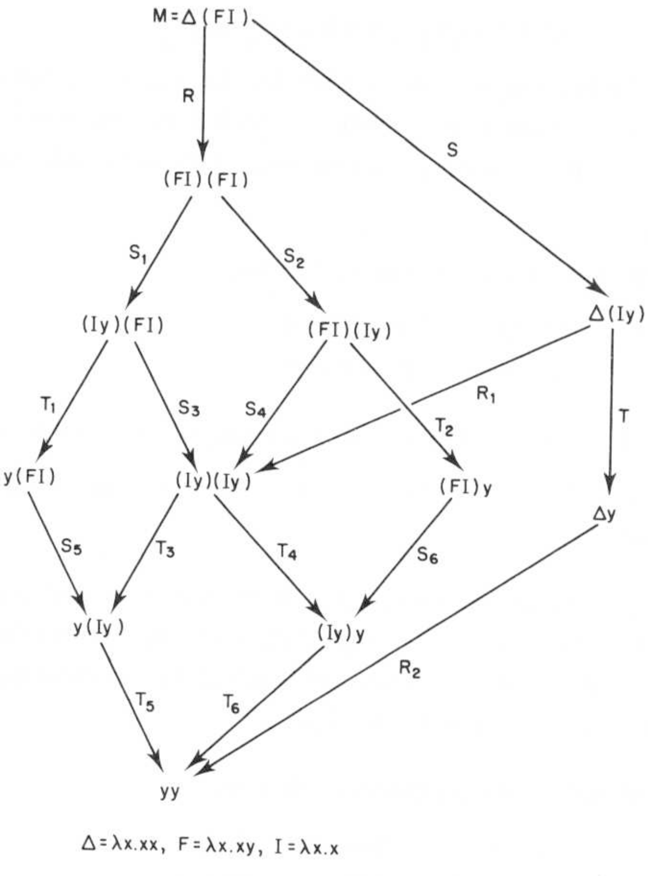
Secondly, the standardisation theorem in [Cu] says that any reduction may be reordered in a standard way. That is : to any reduction corresponds some outside-in and left-to-right reduction. This theorem can be improved. So, let $\rho=R_1R_2\ldots R_n$ be a <u>standard reduction</u> iff, for all i and j such that $1 \le i < j \le n$, $R_j \in R_i'/R_iR_{i+1}\ldots R_{j-1}$ implies that R_i' is internal to R_i or disjoint to the right of R_i. Then, the new standardisation theorem says that, for every reduction ρ, there is a <u>unique</u> standard reduction σ such that $\rho \equiv \sigma$. The existence part of the proof follows from the usual proof of the standardisation theorem which consists in permuting reduction ρ. The unicity comes from remarking that, if $S\rho$ is a standard reduction and R is outside S or to the left of S, then $R/S\rho \neq \emptyset$. Thus standard reductions play the role of canonical representative in permutation-equivalence classes. (The situation is a bit analogous to the uniqueness of left derivations for a given parse tree in context-free formal languages.)

4. DUPLICATIONS OF REDEXES - REDEX FAMILIES :

In the introduction, the interest of looking at duplications of redexes modulo permutations was already mentioned. The example (i) is more completely described in figure 4. There seem to be three kind of redexes in this example. For R and S, it is very easy because all the R_i and S_i's are residuals of R and S. But, for the T case, the only feasible connexion between the T_i's can be exhibited by closing residuals downwards. For instance, the only way of connecting T_1 and T_2 is to say that $T_3 \in T_1/S_3$, $T_3 \in T/R_1$, $T_4 \in T/R_1$ and $T_4 \in T_2/S_4$. Thus, only using residuals, it seems possible to connect even redexes which are not in the initial expression. However, we need to be careful.

Remark first that T_1 and T_2 are not connected if the initial expression is (FI)(FI). Thus the wanted relation needs to be relativized to the initial expression. This can be achieved by considering redexes and their history, i.e. the reductions which give rise to them. In that order, we allow to also read reduction ρR as redex(occurrence) R with <u>history</u> ρ.

Secondly, we must not forget permutations when closing down the residual relation. Otherwise, we could connect R and S in the example of figure 3, which should not be related since, for instance, the Wadsworth's evaluator never contracts them simultaneously. Similarly, if $\Delta = \lambda x.xx$ and $R = \Delta\Delta$, then R with the empty history 0 must not be connected to R with history R.



Δ=λx.xx, F=λx.xy, I=λx.x

Figure 4

This leads us to the following definitions.

Definition 4.1. Redex S with history σ is a copy of redex R with history ρ, written ρR≤σS, iff there is a reduction τ such that ρτ≡σ and S∈R/τ. Similarly, two redexes R and S with histories ρ and σ are in a same family, written ρR≃σS, iff ρR≤σS or σS≤ρR or there is some τT such that ρR≃τT≃σS.

Thus, in the example of figure 4, one gets $RS_1T_1 \simeq RS_2T_2$ since :

$$RS_1T_1 \leq RS_1S_3T_3 \geq ST \leq RS_2S_4T_4 \geq RS_2T_2 \ .$$

Similarly, one can check that $R \not\simeq S$ in figure 3. Now, we study properties of copies and families. Some first easy set of propositions can again be proved by some pure algebraic manipulations.

LEMMA 4.2. Let $\rho \equiv \rho'$ and $\sigma \equiv \sigma'$. Then

1) $\rho R \leq \sigma S$ iff $\rho' R \leq \sigma' S$,
2) $\rho R \simeq \sigma S$ iff $\rho' R \simeq \sigma' S$.

Proof : obvious since $\equiv$ is a congruence for composition. □

LEMMA 4.3. $\rho R \leq \sigma S$ iff $\rho \sqsubseteq \sigma$ and $S \in R/(\sigma/\rho)$. Thus $\leq$ is easily decidable.

Proof : As $\rho R \leq \sigma S$, there is τ such that $\rho\tau \equiv \sigma$ and $S \in R/\tau$. By definition $\rho \sqsubseteq \sigma$. Thus $\rho \sqcup \sigma \equiv \sigma$ and $\rho(\sigma/\rho) \equiv \rho\tau$. By left-cancellation $\sigma/\rho \equiv \tau$. Therefore $R/(\sigma/\rho) = R/\tau$ and $S \in R/(\sigma/\rho)$. Conversely, if $\rho \sqsubseteq \sigma$ and $S \in R/(\sigma/\rho)$, one takes $\tau = (\sigma/\rho)$. □

LEMMA 4.4. $\leq$ is a preorder. Namely,

1) $\rho R \leq \sigma S \leq \tau T$ implies $\rho R \leq \tau T$,
2) $\rho R \leq \sigma S \leq \rho R$ iff $\rho \equiv \sigma$ and $R = S$.

Proof : Since $\rho R \leq \sigma S \leq \tau T$, there are ρ' and σ' such that $\rho\rho' \equiv \sigma$, $\sigma\sigma' \equiv \tau$, $S \in R/\rho'$ and $T \in S/\sigma'$. Thus $\rho\rho'\sigma' \equiv \tau$ and $T \in R/(\rho'\sigma')$. Therefore $\rho R \leq \tau T$. Now suppose $\rho R \leq \sigma S \leq \rho R$. Then $\rho \sqsubseteq \sigma \sqsubseteq \rho$ and $S \in R/(\sigma/\rho)$ by previous lemma. Thus $\rho \equiv \sigma$ and $\sigma/\rho = \emptyset^n$ for $n = |\sigma|$. But $R/\emptyset^n = \{R\}$ and $S = R$. The converse is obvious. □

LEMMA 4.5. $\leq$ satisfies some interpolation and unicity properties:

1) If $\rho \sqsubseteq \sigma \sqsubseteq \tau$ and $\rho R \leq \tau$, there is some redex S such that $\rho R \leq \sigma S \leq \tau T$,

2) If $\rho R_1 \leq \sigma S$ and $\rho R_2 \leq \sigma S$, then $R_1 = R_2$ (same occurrences).

Proof : Let $\rho \sqsubseteq \sigma \sqsubseteq \tau$ and $\rho R \leq \tau T$. Then, by definition, there are reductions ρ', σ', τ' such that $\rho\rho' \equiv \sigma$, $\sigma\sigma' \equiv \tau$, $\rho\tau' \equiv \tau$ and $T \in R/\tau'$. Thus $\rho\tau' \equiv \rho\rho'\sigma'$ and $\tau' \equiv \rho'\sigma'$ by left-cancellation. Thus $R/\tau' = R/(\rho'\sigma')$ and $T \in S/\sigma'$ for some S such that $S \in R/\rho'$. Therefore $\rho R \leq \sigma S \leq \tau T$.

Now, we remark first that the residual definition implies that there is at most one redex R such that $S \in R/\rho$ for given S and ρ. Thus, if $\rho R_1 \leq \sigma S$ and $\rho R_2 \leq \sigma S$, we have $S \in R_1/(\sigma/\rho)$ and $S \in R_2/(\sigma/\rho)$. Therefore $R_1 = R_2$. □

We turn now to the hard part of this paper, which is to show that the family relation is decidable. The trouble comes from the necessity of looking now inside λ-expressions and from not being able to go on with algebraic manipulations. But first of all there is an easy case when one considers families of redexes with an empty history.

LEMMA 4.6. $R \simeq \rho S$ iff $S \in R/\rho$.

Proof : First, if $S \in R/\rho$, then $R \leq \rho S$ and $R \simeq \rho S$. Conversely, we use an induction on the recursive definition of $\simeq$. Thus, we assume that there is some τT such that $R \simeq \tau T$, $T \in R/\tau$ and either one of the following two cases. First $\tau T \leq \rho S$. Then $R \leq \rho S$ since $R \leq \tau T$. And $S \in R/\rho$, since $\rho = \rho/0$. Otherwise $\rho S \leq \tau T$. Then $\rho \sqsubseteq \tau$. Since $0 \sqsubseteq \rho \sqsubseteq \tau$, we get by interpolation $R \leq \rho S' \leq \tau T$ for some redex S'. But $S = S'$ by unicity. Therefore $R \leq \rho S$ and $S \in R/\rho$. □

Roughly speaking, the decision procedure for the family relation is as follows : $\rho R \simeq \sigma S$ iff R and S are "created" in the same way along ρ and σ, when ρ and σ are standard reductions. The problem is to formalise creations of redexes. This is not easy in the λ-calculus. One way is to define a labelled λ-calculus (see [Le2]) following the idea of [Vu] for recursive programs schemes. Another way, considered here and in [Bel], introduces an extraction operation on reductions and is to characterize redexes which do not create R in ρR.

Let a context C[ ] be a λ-expression with some missing subexpression and C[M] be the λ-expression obtained by filling the hole by the expression M. Similarly, we may have contexts C[, , ...] with several (disjoint) holes. Let two reductions ρ and σ starting at M be disjoint iff they are internal to two disjoint subexpressions of M. This means that M=C[N,P] and $\rho:C[N,P] \overset{*}{\to} C[N',P]$, $\sigma:C[N,P] \overset{*}{\to} C[N,P']$.

Let the function part of redex R=(λx.M)N be the left subexpression (λx.M), and the argument part of R the right subexpression N.

Suppose now x is a free variable in M. Let M^x[, , ...] be the context corresponding to M without all free occurrences of x. Assume that the reduction Rρ is such that ρ is internal to the $i^{\underline{th}}$ instance of the argument N of R=(λx.M)N in its contractum M[x\N]. Thus Rρ is as in figure 5 (d). Let then the reduction ρ//R, read ρ parallelised by R, be the reduction defined inductively by :

$$0//R=0$$

$$(S\rho)//R=(S'/R)((\rho/F)//(R/S')) \text{ where } S\in S'/R \text{ and } F=S'/(RS).$$

Remark that, since Sρ is in the $i^{\underline{th}}$ instance of the argument of R in its contractum, one has S' in the argument part of R, F disjoint from ρ, R/S'={R_1} and ρ/F in the $i^{\underline{th}}$ instance of the argument of R_1 in its contractum. (See again figure 5 (d)).

Now, we eliminate unnecessary steps of ρ for redex R with history ρ.

Definition 4.7. The extraction relation ⊳ is the union of the four following relations :

1) $\rho RS \triangleright_1 \rho S'$ if $S \in S'/R$,

2) $\rho(R \sqcup \sigma) \triangleright_2 \rho\sigma$ if $|\sigma| \geq 1$ and R, σ are two disjoint reductions,

3) $\rho(R \sqcup \sigma) \triangleright_3 \rho\sigma$ if $|\sigma| \geq 1$ and σ is a reduction internal to the function part of R,

4) $\rho R\sigma \triangleright_4^i \rho\sigma'$ if $|\sigma| \geq 1$, σ is internal to the i^{th} instance of the argument of R in its contractum and $\sigma'/R = \sigma//R$.

This definition is summarized in figure 5. Remark that, in the last case, the reduction σ' is in the argument part of R. For instance, in the example of figure 4, one has :

$$RS_1S_3 \triangleright RS_2 \triangleright S,$$
$$S\ T\ R_2 \triangleright SR_1 \triangleright R,$$
$$RS_1T_1 \triangleright ST.$$

Let $\trianglerighteq$ denote the transitive closure of $\triangleright$, i.e. $\rho \trianglerighteq \sigma$ iff there is a (possibly empty) chain of extractions leading from ρ to σ. The key property is the following lemma.

LEMMA 4.8. $\trianglerighteq$ has the Church-Rosser property, i.e. if $\rho \triangleright \sigma$ and $\rho \triangleright \tau$, then $\sigma \triangleright \nu$ and $\tau \triangleright \nu$ for some reduction ν.

The proof is tedious, as usual for Church-Rosser properties, because of its number of cases, and is sketched in the appendix. It relies mainly on the following remark : if R and S are two distinct redexes in an expression M, and T is a redex such that $T \in T_1/(R/S)$ and $T \in T_2/(S/R)$, then there is some T' such that $T \in T'/(R \sqcup S) = T'/(S \sqcup R)$. Now comes the decision procedure for the family relation.

THEOREM 4.9. Let ρ and σ be two standard reductions. Then $\rho R \simeq \sigma S$ iff $\rho R \trianglerighteq \tau T \trianglelefteq \sigma S$ for some τT.

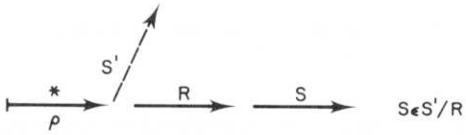
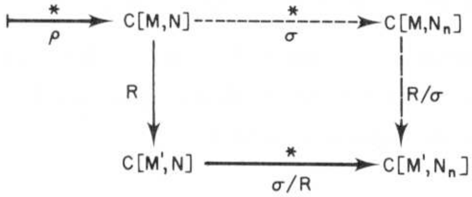
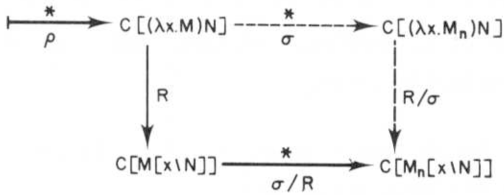


Figure 5 (a)



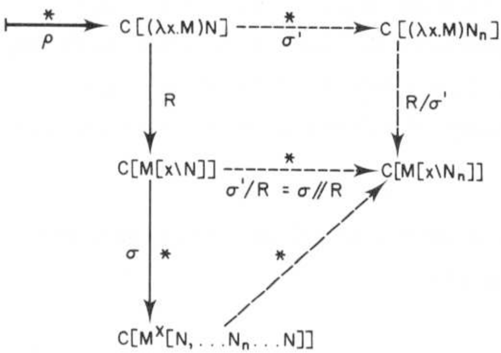
R in M and σ in N

Figure 5 (b)



σ in N

Figure 5 (c)



σ in one instance of N in M [x\N]

Figure 5 (d)

Proof : Remark first that $\rho R \triangleright \tau T$ always implies $\rho R \simeq \tau T$. In the first three cases of the definition of $\triangleright$, one gets obviously $\tau T \leq \rho R$. In the last case, if $\rho = \rho' R' \rho''$, then $\tau = \rho' \tau'$ and $\rho'' \nu \equiv \tau' / R'$ where $\nu = \tau' / (R' \rho'')$. Furthermore, $R/\nu = \{T'\}$ and $T' \in T/(R'/\tau')$. Thus $\rho R \leq \rho'(R' \sqcup \tau')T'$ and $\rho'(R' \sqcup \tau')T' \geq \tau T$. Therefore, when $\rho R \trianglerighteq \tau T$ and $\sigma S \trianglerighteq \tau T$, one gets $\rho R \simeq \sigma S$.

Conversely, it is enough to show that $\rho R \leq \sigma S$ implies $\rho R \trianglerighteq \tau T \trianglelefteq \sigma S$ for some τT, because of the Church-Rosser property of $\triangleright$. For suppose ρ and σ are standard reductions and $\rho R \simeq \sigma S$. Then there is a chain of $\rho_i R_i$ such that $\rho_0 R_0 = \rho R$ and $\rho_n R_n = \sigma S$ and, for $1 \leq i \leq n$, either $\rho_{i-1} R_{i-1} \leq \rho_i R_i$ or $\rho_i R_i \leq \rho_{i-1} R_{i-1}$. By lemma 4.2, one can always assume that ρ_i is standard. We shall prove the existence of $\tau_i T_i$ for $1 \leq i \leq n$ such that $\rho_{i-1} R_{i-1} \trianglerighteq \tau_i T_i \trianglelefteq \rho_i R_i$. Then by Church-Rosser of $\triangleright$, we may conclude that there is some τT such that $\rho R \trianglerighteq \tau T \trianglelefteq \sigma S$.

So let ρ and σ be standard reductions and $\rho R \leq \sigma S$. We use an induction on $|\sigma|$. If $\rho = 0$, then $\sigma/\rho = \sigma$ and $S \in R/\sigma$. Thus $\sigma S \trianglerighteq \rho R$ and $\rho R \trianglerighteq \rho R$. Let $\rho \neq 0$. Then, as $\rho R \leq \sigma S$ implies $\rho [\sigma$, one cannot have $\sigma = 0$. Thus $\sigma = S' \sigma'$. Let too $\rho = R' \rho'$. If $R' = S'$, then $\rho' R \leq \sigma' S$, since $[$ is left-cancellable. By induction, there is $\tau' T$ such that $\rho' R \trianglerighteq \tau' T$ and $\sigma' S \trianglerighteq \tau' T$. Thus, if $\tau = R' \tau'$, we get $\rho R \trianglerighteq \tau T \trianglelefteq \sigma S$. Suppose now $R' \neq S'$, which is in fact the only interesting case. Then, since $\rho [\sigma$, R' cannot be external to S' or to its left. Otherwise $R'/\sigma \neq \emptyset$ which contradicts $\rho/\sigma = \emptyset^n$. Thus, since ρ is standard, there is a decomposition of $\rho = \rho_f \sqcup \rho_a \sqcup \rho_d$ such that ρ_f, ρ_a, ρ_d are standard reductions respectively internal to the function part of S', internal to the argument part of S' and disjoint from S'. (Remark that the definition of $\sqcup$ makes it associative). Now we have several cases with respect to the relative positions of R and of the residual S'' of S' by ρ.

1) If R is external to S'' or to the left of S''. Then one proves easily by induction on $|\rho|$ that $R \in T/\rho$ for some redex T external to S' or to the left of S'. Thus $\rho R \trianglerighteq T$. But, since $\rho R \leq \sigma S$,

one gets $T \simeq \sigma S$ and $S \in T/\sigma$ by lemma 4.5. Therefore $\sigma S \trianglerighteq T$.

2) If R is internal to the function part of S'', let then $\{S_f\} = S'/\rho_f$ and $\nu = (\rho_a \sqcup \rho_d)/\rho_f$. Then $S_f/\nu = \{S''\}$ and the reduction ν is internal to the argument of S_f or disjoint from S_f. Thus ν is disjoint from the function part of S_f. Thus there is a redex R_f in the function part of S_f such that $R_f/\nu = \{R\}$. Therefore $\rho R \trianglerighteq \rho_f R_f$ and $\rho R \geq \rho_f R_f$. Since $\rho R \leq \sigma S$, one gets too $\rho_f R_f \leq \sigma S$. By lemma 4.3, $S \in R_f/(\sigma/\rho_f)$. So, if $(\rho_f R_f)/S' = \rho_f' R_f'$, then $\rho_f' R_f' \leq \sigma' S$ and $S' \rho_f' R_f' \triangleright \rho_f R_f$. Now, since ρ_f is in the function part of S' and ρ_f is standard, the reduction $\rho_f' = \rho_f/S'$ is also standard. Thus, by induction, there is $\tau' T'$ such that $\sigma' S \trianglerighteq \tau' T'$ and $\rho_f' R_f' \trianglerighteq \tau' T'$. Therefore $\sigma S \trianglerighteq S' \tau' T'$ and $S' \rho_f' R_f' \trianglerighteq S' \tau' T'$. By Church-Rosser for $\trianglerighteq$, there is some τT such that $S' \tau' T' \trianglerighteq \tau T$ and $\rho_f R_f \trianglerighteq \tau T$. Thus $\sigma S \trianglerighteq \tau T$ and $\rho R \trianglerighteq \tau T$.

3) If R is disjoint from S'' and to the right of S'', there is again some R_d, disjoint from the residual S_d of S' by ρ_d, such that $\rho R \trianglerighteq \rho_d R_d$ and $\rho R \geq \rho_d R_d$. Then one goes on as previously.

4) If R is in the argument part of S'', then there is too some redex R_a in the argument part of the residual S_a of S' by ρ_a such that $\rho R \trianglerighteq \rho_a R_a$ and $\rho R \geq \rho_a R_a$. We do as previously, but one gets trouble, since ρ_a/S' is no longer some standard reduction. However $\rho_a R_a \leq \sigma S$, since $\rho R \leq \sigma S$. Thus, by lemma 4.3, $S \in R_a/(\sigma/\rho_a)$. So, there is some redex $R_a'' \in R_a/S_a$ such that $\rho_a'' R_a'' \leq \sigma' S$, where $\rho_a'' = \rho_a/S'$. But R_a'' is in some instance of the argument of S_a in its contractum. Say the $i^{\underline{th}}$ instance. Then, since ρ_a'' is the union of disjoint reductions, each of them being internal to some instance of the argument of S' in its contractum, if ρ_a' is the part of ρ_a'' inside the $i^{\underline{th}}$ instance, then there is R_a' such that $\rho_a' R_a' \leq \rho_a'' R_a''$ and $S' \rho_a' R_a' \triangleright \rho_a R_a$. Since $\rho_a'' R_a'' \leq \sigma' S$, one has too $\rho_a' R_a' \leq \sigma' S$. But, now ρ_a' is standard, and we can go on by induction. So there is $\tau' T'$ such that $\sigma' S \trianglerighteq \tau' T'$ and $\rho_a' R_a' \trianglerighteq \tau' T'$. Thus $\sigma S \trianglerighteq S' \tau' T'$ and $S' \rho_a' R_a' \trianglerighteq S' \tau' T'$. By Church-Rosser of $\trianglerighteq$, there is τT such that $S' \tau' T' \trianglerighteq \tau T$ and $\rho_a R_a \trianglerighteq \tau T$. Therefore $\sigma S \trianglerighteq \tau T$ and $\rho R \trianglerighteq \tau T$. □

Remark that the previous theorem really gives a decision procedure since, when ρR and σS are given, there are effective ways of finding the standard reductions ρ' and σ' equivalent to ρ and σ by permutations (see [K1]) and furthermore the extraction relation $\trianglerighteq$ always ends at some normal form (because $\rho R \triangleright \tau T$ implies $|\rho|>|\tau|$).

Notice too that, if ρ is a standard reduction and $\rho R \triangleright \sigma S$, σ is also standard. This is obvious by considering the definition of $\triangleright$. Therefore, we can conclude that, in each redex family class, there is only one ρR such that ρ is standard and $|\rho|$ is minimum. Similarly, this ρR is the only one such that ρ is standard and ρR is in $\triangleright$-normal form. This canonical representative of the family class of, say, σS will be written $e^{*}(\sigma S)$ and named the <u>normal form by extractions</u>. Thus $\sigma_s S \trianglerighteq e^{*}(\sigma S)$ where σ_s is the standard reduction such that $\sigma_s \equiv \sigma$.

Finally, take the example of figure 4. Then there are three family classes with canonical representatives R,S,ST. In general, if $R \simeq \rho S$, then $R=e^{*}(\rho S)$ by lemma 4.6. This lemma has a nice generalisation, which will be fundamental in the rest of the paper. We first remark that the extraction relation can be done in some right to left order.

LEMMA 4.10. If $R\rho \triangleright \sigma$ and $\sigma \triangleright \tau$, there is some ν such that $\rho \triangleright \nu$ and $R\nu \triangleright \tau$.

The proof is similar to the proof of 4.6.

LEMMA 4.11. Let $\rho R=e^{*}(\sigma S)$. Then $\rho \sqsubseteq \sigma$ iff $\rho R \leq \sigma S$.

Proof : The if-part follows from 4.3. Suppose now that τ is the standard reduction such that $\tau \equiv \sigma$. We have $\tau S \trianglerighteq \rho R$ and $\rho \sqsubseteq \tau$, since $\rho R=e^{*}(\sigma S)$ and $\rho \sqsubseteq \sigma$. Moreover $\tau S=e^{*}(\tau S)$ is in normal form with respect to extractions.

By 4.2, it is sufficient to show $\rho R \leq \tau S$, since $\tau \equiv \sigma$. We work by induction on $|\tau|$. If $\tau=0$, then $\tau S \trianglerighteq \rho R$ implies $\rho=0$ and $R=S$. Thus $\rho R \leq \tau S$. Let now $\tau=T\tau'$. By using the previous lemma, we have only two cases :

1) $\rho=T\rho'$ and $\rho' R=e^{*}(\tau' S)$. Then $\rho \sqsubseteq \tau$ implies $\rho' \sqsubseteq \tau'$ by left-cancellation. By induction $\rho' R \leq \tau' S$. And thus $\rho R \leq \tau S$.

2) $\rho' R'=e^{*}(\tau' S)$ and $T\rho' R' \triangleright \rho R$. Then by the definition of $\triangleright$, one always has $\rho' \sqsubseteq \rho/T$. Since $\rho \sqsubseteq \tau$, then $\rho/T \sqsubseteq \tau/T$. But $\tau/T=\emptyset\tau' \equiv \tau'$. Thus $\rho' \sqsubseteq \rho/T \sqsubseteq \tau'$. Therefore, by induction, $\rho' R' \leq \tau' S$. And, by interpolation, there is some redex S' such that $\rho' R' \leq (\rho/T) S' \leq \tau' S$. Now, we again look at the definition of $\triangleright$. Let $\rho''=\rho/(T\rho')$. Then ρ'' is always disjoint from R' and there is only one residual R'' of R' by ρ''. Furthermore $\rho R \leq (T \sqcup \rho) R''$. But, by 4.3, $S' \in R'/\rho''$. Thus $S'=R''$. And $\rho R \leq (T \sqcup \rho) S' \leq T\tau' S$. That is $\rho R \leq \tau S$. □

5. COMPLETE REDUCTIONS

First, we generalise the finite developments theorem. Let $[\rho R]$ be the family class of ρR, i.e. the equivalence class of ρR with respect to $\simeq$. Let $FAM(\rho)$ be the set of family classes of the redexes contracted in ρ. More exactly, if $\rho=\mathcal{F}_1\mathcal{F}_2\ldots\mathcal{F}_n\ldots$,

$$FAM(\rho)=\{[\mathcal{F}_1\mathcal{F}_2\ldots\mathcal{F}_{n-1}R_n] \mid R_n \in \mathcal{F}_n,\ n\geq 1\}$$

Say that ρ is <u>relative</u> to X if $FAM(\rho)\subset X$. Similarly, a reduction ρ relative to X is a <u>development</u> of X if there is no redex R such that $[\rho R]\in X$. Then lemma 4.6 tells us that these definitions are exact extensions of the ones of §2.

THEOREM 5.1. (Generalised finite developments theorem). Let X be a finite set of family classes. Then :

1) there is no infinite reduction relative to X,

2) if ρ and σ are two developments of X, then $\rho\equiv\sigma$. (This implies that ρ and σ end at the same expression and $\tau/\rho=\tau/\sigma$ for for all reductions τ starting at the initial expression of ρ and σ.)

Proof : The finiteness part is proved by using a labelled λ-calculus (see [Le1] with a so-called bounded predicate. In that case, there is a strong normalisation property of this calculus. And it is straightforward to show that relative reductions to some finite set X can be embedded in such a calculus. (One has just to show that $S \in R/\rho$ implies that R and S have the same labels. Thus $\rho R \simeq \sigma S$ also implies that R and S have the same labels). Now the second part of the theorem follows easily from noticing that, when ρ and σ are relative to X, also $\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are relative to X. □

Developments have nice properties with respect to the family relation. This follows from the two next remarks. First, when $\sigma S \triangleright \rho R$, one gets clearly $FAM(\rho) \subset FAM(\sigma)$. Secondly, also $FAM(\rho) \subset FAM(\sigma)$, when ρ is the standard reduction such that $\rho \equiv \sigma$. (This comes directly from the proof of the standardisation theorem). Thus, if $\rho R = e^*(\sigma S)$, the reduction ρ is relative to FAM(σ).

Notice too that, when ρ is relative to X and σ is a development of X, one has $\rho \sqsubseteq \sigma$, since one always can extend ρ to some development ρτ of X and $\rho\tau \equiv \sigma$ by the previous theorem. Therefore, one gets the two following lemmas.

LEMMA 5.2. Let ρ be a development of X. Then, for all redexes R with history ρ, one has $e^*(\rho R) \leq \rho R$.

Proof : Let $\sigma S = e^*(\rho R)$. Then σ is relative to FAM(ρ). Therefore σ is relative to X, since $FAM(\rho) \subset X$. Thus $\sigma \sqsubseteq \rho$, since ρ is a development of X. By 4.11, $\sigma S \leq \rho R$. □

LEMMA 5.3. Let ρ be a development of X. Then, for any σS such that $\rho \sqsubseteq \sigma$, one has $[\sigma S] \notin FAM(\rho)$.

Proof : First, since ρ is a development of X, ρ is also a development of FAM(ρ). Thus there is no T such that $[\rho T] \in$ FAM(ρ). Now, suppose $\rho \sqsubseteq \sigma$ and $[\sigma S] \in$ FAM(ρ). Then $\rho = \rho_1 \mathcal{F} \rho_2$ and there is some redex $R \in \mathcal{F}$ such that $\rho_1 R \simeq \sigma S$. Let $\rho' R' = e^*(\rho_1 R)$. Then $e^*(\rho_1 R) = e^*(\sigma S)$, and ρ' is relative to FAM(ρ_1). Thus ρ' is relative to FAM(ρ) and $\rho' \sqsubseteq \rho$. By 4.11, $\rho' R' \leq \sigma S$. Therefore $\rho' R' \leq \rho T \leq \sigma S$ by interpolation for some redex T. Thus $\rho T \simeq \rho' R' \simeq \rho_1 R$ and $[\rho T] \in$ FAM(ρ). Contradiction. □

Now, we consider complete reductions, which we will show as being particular developments. Let a reduction $\mathcal{F}_1 \mathcal{F}_2 \ldots \mathcal{F}_n \ldots$ be complete iff, for every $n \geq 1$, $\mathcal{F}_n \neq \emptyset$ is a maximum set of redexes such that, for all $R \in \mathcal{F}_n$ and $S \in \mathcal{F}_n$, $\mathcal{F}_1 \mathcal{F}_2 \ldots \mathcal{F}_{n-1} R \simeq \mathcal{F}_1 \mathcal{F}_2 \ldots \mathcal{F}_{n-1} S$.

Thus, at each step of a complete reduction, one non-empty family class is contracted.

LEMMA 5.4. Any complete reduction ρ is a development of FAM(ρ).

Proof : By induction on $|\rho|$. The case $\rho = 0$ is obvious. Let $\rho = \sigma \mathcal{F}$. By induction σ is a development of FAM(σ). Thus there is no ρR such that $[\rho R] \in$ FAM(σ) by 5.3, since $\sigma \sqsubseteq \rho$. Suppose now that there is R and $S \in \mathcal{F}$ such that $\rho R \simeq \sigma S$. Then $e^*(\sigma S) \leq \sigma S$ by 5.2. Let $\sigma' S' = e^*(\sigma S)$. Then $\sigma' \sqsubseteq \sigma \sqsubseteq \rho$ by 4.3. By 4.11, since also $\sigma' S' = e^*(\rho R)$, one gets $\sigma' S' \leq \rho R$. By interpolation, there is some T such that $\sigma' S' \leq \sigma T \leq \rho R$. Therefore $\sigma S \simeq \sigma T$ and $T \in \mathcal{F}$, since ρ is complete. By 4.3, $R \in T/(\rho/\sigma)$. Therefore $R \in T/\mathcal{F}$ and $R \in \mathcal{F}/\mathcal{F}$, since $T \in \mathcal{F}$. Contradiction since $\mathcal{F}/\mathcal{F} = \emptyset$. □

LEMMA 5.5. Let ρ be a complete reduction. Then $|\rho| = \#$FAM(ρ), where $\#$FAM(ρ) is the number of elements in FAM(ρ).

Proof : Let $\rho=F_1F_2\ldots F_n$. Then $|\rho|=n$. Let $1\leq i<j\leq n$ and $\rho_i=F_1F_2\ldots F_{i-1}$, $\rho_j=F_1F_2\ldots F_{j-1}$. Then, since ρ_i is complete and $\rho_i\sqsubseteq\rho_j$, one cannot have $\rho_i R\simeq\rho_j S$ for some $R\in F_i$ and $S\in F_j$, by using 5.4 and 5.3. Furthermore $F_i\neq\emptyset$ for every F_i. Thus $n=\#\mathrm{FAM}(\rho)$. □

Further properties of complete reductions may be shown (see [Be1,Le2]. For instance, complete reductions make a sub-semi-lattice of the one of reductions. That is $\rho\sqcup\sigma$ is complete (up to some empty steps), when ρ and σ are complete reductions.

6. OPTIMAL REDUCTIONS

We come back to the problem discussed in the introduction. First, we show that call-by-need stategies are terminating. Let some reduction ρ be terminating iff its final expression is in normal form. Now, if $\rho=F_1F_2\ldots F_n\ldots$, let $R(\rho)$ be the set of redexes one of whose residuals is contracted in ρ. More exactly,

$$R(\rho)=\{R\,|\,R/F_1F_2\ldots F_{i-1}\cap F_i\neq\emptyset,\ i\geq 1\}$$

Let some redex R in an expression M be needed iff, for all terminating reductions ρ starting at M, one has $R\in R(\rho)$. Let a reduction $\rho=F_1F_2\ldots F_n\ldots$ be a call-by-need reduction, iff there is at least one needed redex in every F_n, for $n\geq 1$.

We easily show that there is at least a needed redex in any expression M. Take R as being the leftmost-outermost redex of M. Let $\rho:M\overset{*}{\rightarrow}N$ be a reduction such that $R\notin R(\rho)$. Then $R/\rho=\{S\}$ and S is the leftmost-outermost redex in N. Thus R is needed, since, for all terminating ρ issued from M, one has $R\in R(\rho)$, since one must have $R/\rho=\emptyset$.

THEOREM 6.1. Let M have a normal form. Then any call-by-need reduction starting at M is eventually terminating.

Proof : By the standardisation theorem [Cu], since M has a normal form, the leftmost-outermost reduction eventually reaches the normal form. Let d(M) be the length of the terminating leftmost-outermost reduction issued from M. Let

$$\sigma: M \xrightarrow{F_1} M_1 \xrightarrow{F_2} M_2 \;\dots\; \xrightarrow{F_n} M_n \xrightarrow{F_{n+1}} \dots$$

be a call-by-need reduction. We want to show that

$$d(M) > d(M_1) > d(M_2) > \dots d(M_n) > \dots$$

Thus, we will have $d(M_p)=o$ for some $p \geq 1$. That is M_p in normal form.

So let $\rho = R_1 R_2 \dots R_n$ be the terminating leftmost-outermost reduction starting at M. Then $\rho / F_1 = C_1 C_2 \dots C_n$ with, for every i, either $C_i = \emptyset$ or $C_i = \{S_i\}$ where S_i is leftmost-outermost (since residuals of leftmost-outermost redexs remain leftmost-outermost). Thus ρ/F_1 is the terminating leftmost-outermost reduction issued from M_1 (up to some empty steps). Now, as σ is call-by-need reduction, there is a needed redex $T \in F_1$. Therefore $T \in R(\rho)$. That is $R_i \in T/R_1 R_2 \dots R_{i-1}$ for some i. Thus $C_i = \emptyset$ and $d(M) > d(M_1)$. Similarly $d(M_1) > d(M_2)$, etc... □

Now, we define some cost measures for reductions. As said in the introduction, some λ-evaluator using sharing allows to contract copies of a single redex in one unit of time. So let us say that F with history ρ is a <u>set of copies of a single redex</u> iff there is one redex S with history σ such that $\sigma S \leq \rho R$ for every $R \in F$. Natural reductions to consider are now the following c-complete reductions. Say that the reduction $\rho = F_1 F_2 \dots F_n \dots$ is <u>c-complete</u> iff, for all $n \geq 1$, the non-empty set F_n is a maximum set of copies of a single redex. Now, we shall assume that the cost of such a c-complete reduction satifies the equation : $cost(\rho) = |\rho|$.

Furthermore, since all redexes in a set of copies of a single redex are in a same redex family and since we do not contract in one unit of time redexes which are not copies of one redex, we may assume for any reduction ρ: $cost(\rho) \geq \# FAM(\rho)$.

We first show that these two constraints one the cost measure are compatible and that c-complete reductions are effective, because they correspond exactly to complete reductions. (Remark that c-complete reductions are the right version of non-copying reductions in [O'D].)

LEMMA 6.2. A reduction is c-complete iff it is a complete reduction.

Proof : Suppose ρ is complete. Let R be redex with history ρ. Let F be the set of redexes S such that $\rho R \simeq \rho S$ and F' a maximum set such that $R \in F'$ and there is some σT such that $\sigma T \leq \rho S$ for all $S \in F'$. We want to show $F = F'$. First $F' \subset F$ since, for all $S \in F'$ and $S' \in F'$, we have $\sigma T \leq \rho S$ and $\sigma T \leq \rho S'$. Therefore $\rho S \simeq \rho S'$. Now by 5.4 and 5.2, since ρ is a complete reduction, ρ is a development of FAM(ρ). Therefore $e^{*}(\rho S) \leq \rho S$ for every $S \in F$. But, for all $S \in F$ and $S' \in F$, since $\rho S \simeq \rho S'$, we have $e^{*}(\rho S) = e^{*}(\rho S')$. Therefore F is a set of copies of a single redex. Since $R \in F$, we thus get $F \subset F'$, since F' is maximum.

Now, we prove easily by induction on $|\rho|$ that a reduction ρ is complete iff it is c-complete. The case $\rho = 0$ is obvious. Let now $\rho = \sigma F$. By induction σ is complete iff σ is c-complete. Now, if ρ is complete, by the first part of the proof, F is a maximum set of copies of a single redex. Therefore ρ is c-complete. Suppose now ρ c-complete. then σ is complete by induction and F is one family class, by again the first part of this proof. Thus ρ is complete. □

Therefore, we can speak only of complete reductions. And lemma 5.5 tells us that the discussion on the cost measure is consistent, since $|\rho| = \#\text{FAM}(\rho)$ for any complete reduction ρ. Notice too that, if $\rho = R_1 R_2 \ldots R_n \ldots$ is some non parallel reduction, we have too $\#\text{FAM}(\rho) \leq |\rho|$. Now, we prove the optimality theorem.

THEOREM 6.3. Any complete and call-by-need reduction reaches the normal form in an optimal cost.

Proof : Let ρ be a call-by-need and complete reduction. Let σ be a terminating reduction starting at the same expression as ρ. We first prove $FAM(\rho) \subset FAM(\sigma)$. The case $\rho=0$ is trivial. Let $\rho=\rho' F$. Then $FAM(\rho) \subset FAM(\rho') \cup \{[\rho' S]\}$ for any $S \in F$. By induction $FAM(\rho') \subset FAM(\sigma)$. But σ/ρ' is also terminating, since the final expression of σ is in normal form. Since there is some needed redex R in F, we know that $R \cap \tilde{R}(\sigma/\rho') \neq \emptyset$. Therefore $\sigma/\rho' = \sigma_1' F_2' \sigma_3'$ with $R/\sigma_1' \in F_2'$. But $\sigma = \sigma_1 F_2 \sigma_3$ with $\sigma_1' = \sigma_1/\rho'$, $F_2' = F_2/(\rho'/\sigma_1)$. Thus there is $R_2 \in F_2$ and $R_2' \in F_2'$ satisfying :

$$\sigma_1 R_2 \leq (\sigma_1 \sqcup \rho') R_2' \geq \rho' R.$$

Therefore $[\rho' R] \in FAM(\sigma)$ and $FAM(\rho) \subset FAM(\sigma)$. Thus, we may conclude that any terminating call-by-need reduction ρ reaches the normal form in $|\rho|$ steps such that :

$$|\rho| = \# FAM(\rho) \quad \text{(by 5.5)}$$

and

$$|\rho| = cost(\rho) \leq \# FAM(\sigma) \leq cost(\sigma)$$

for any other terminating reduction σ. □

COROLLARY 6.4. The leftmost-outermost complete reduction reaches the normal form in an optimal cost.

Other call-by-need strategies were studied in [Le2] and correspond to the safe computation rules in [Vu]. The converse of theorem 6.3 may also be proved, i.e., among the complete reductions, only the call-by-need ones are optimal.

7. CONCLUSION

It remains to design some λ-evaluator implementing our complete and call-by-need reductions. The trouble, as stressed in [Wa], comes from the bound variables. Our results were expressed in words of term rewriting systems and, thus, seem somewhat general. Therefore, some general theory of rewriting systems, including the λ-calculus case, would be welcomed. For instance in [Hu], the problem of defining call-by-need reductions is considered. However, the most important question is to find shared evaluators for rewriting systems with bound variables.

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9. APPENDIX

Proof of lemma 4.8 : Cases will be number m.n where $\rho \triangleright_m \rho'$ and $\sigma \triangleright_n \sigma'$.

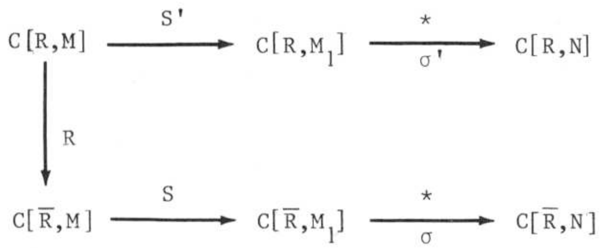
1) $RS \triangleright_1 S'$ with $S \in S'/R$.

1.2) $RS \triangleright_2 S''$ with $S''/R = \{S\}$ and S'' disjoint from R. Then $S'' = S'$.

1.3, 1.4) Similar cases.

Now, we notice that, in case $m \geq 2$ and $|\rho| \geq 1$, $|\sigma| \geq 1$, one has $R\rho\sigma \triangleright_m \rho'\sigma'$ iff $R\rho \triangleright_m \rho'$, σ is disjoint from $\rho'' = \rho'/(R\rho)$ and $(R/\rho)(\sigma/\rho'') \triangleright_m \sigma'$. Remark that, when m = 2 or m = 3, then $\rho = \rho'/R$ and $\rho'' = \emptyset^k$, $\sigma/\rho'' = \sigma$. Furthermore, we also remark that, when $\rho \triangleright \sigma$ and τ is disjoint from ρ, then $\rho/\tau \triangleright \sigma/\tau$ and σ is disjoint from τ. Therefore, it is enough to show that, when $RS\sigma \triangleright S'\sigma'$ and $RS\sigma \triangleright R\tau$, then there is τ' such that $S'\sigma' \triangleright \tau'$ and $R\tau \triangleright \tau'$.

2) $RS\sigma \triangleright_2 S'\sigma'$ with $S'\sigma'$ disjoint from R and $S\sigma = (S'\sigma')/R$. Therefore, the initial expression is of the form C[R,M] and reductions $RS\sigma$ and $S'\sigma'$ are of the form :



Then, when $S\sigma \triangleright_n \tau$, since $S\sigma$ is internal to M, τ is also internal to M. Therefore one checks easily that $R\tau \triangleright \tau'$ and $S'\sigma' \triangleright \tau'$.

3) $RS\sigma \triangleright_3 S'\sigma'$ with $S'\sigma'$ in the function part of R and $S\sigma = (S'\sigma')/R$. Therefore the initial expression is of the form $C[(\lambda x.M)N]$ where $R = (\lambda x.M)N$. Reductions $RS\sigma$ and $S'\sigma'$ are of the form :

$$\begin{array}{ccccc}
C[(\lambda x.M)N] & \xrightarrow{S'} & C[(\lambda x.M_1)N] & \xrightarrow[\sigma']{*} & C[(\lambda x.M_n)N] \\
\downarrow R & & & & \\
C[M[x\backslash N]] & \xrightarrow{S} & C[M_1[x\backslash N]] & \xrightarrow[\sigma]{*} & C[M_n[x\backslash N]]
\end{array}$$

We treat this case more algebraically.

3.1) $\sigma = T$ and $ST \triangleright_1 V$ because $T \in V/S$. But $\sigma' = T'$ and $T \in T'/(R/S')$. Since $\{S\} = S'/R$, one has $T \in V/(S'/R)$. Since $R \neq S'$, there is some V' such that $T \in V'/(R \sqcup S') = V'/(S' \sqcup R)$. Therefore $V \in V'/R$ and $T' \in V'/S'$. Hence $S'T' \triangleright_1 V'$. Furthermore, since $S'T'$ is in the function part of R and since $T' \in V'/S'$, we also have V' in the function part of R. Therefore $RV \triangleright_2 V'$.

3.2) $S\sigma \triangleright_2 \tau$ because τ is disjoint from S and $\tau/S = \sigma$. Again, since $R \neq S$, there is τ' such that $\sigma' = \tau'/S'$ and $\tau = \tau'/R$. Now, one checks easily that τ' is disjoint from S' and in the function part of R. Therefore $S'\sigma' \triangleright_2 \tau'$ and $R\tau \triangleright_3 \tau'$.

3.3) $S\sigma \triangleright_3 \tau$ because τ is in the function part of S and $\tau/S = \sigma$. This case is similar to the previous one.

3.4) $S\sigma \triangleright_4^i \tau$ because τ is in the $i^{\underline{th}}$ instance of the argument of S in its contractum and $\sigma /\!/ S = \tau/S$. This case is similar to the previous ones in case $|\sigma| = 1$. Now, when $\sigma = \sigma_1\sigma_2$, we use the decomposition of $S\sigma_1\sigma_2 \triangleright_4^i \tau_1\tau_2$ according to the remark preceeding case 2.

4) $RS\sigma \triangleright_4^i S'\sigma'$ with $S\sigma$ in the $i^{\underline{th}}$ instance of the argument of R in its contactiun and $(S'\sigma')/R = (S\sigma) /\!/ R$. Then the initial expression of $RS\sigma$ is of the form $C[(\lambda x.M)N]$ and $R = (\lambda x.M)N$. Then $RS\sigma$ and $S'\sigma'$ are such that

$$
\begin{array}{ccccc}
C[(\lambda x.M)N] & \xrightarrow{S'} & C[(\lambda x.M)N_1] & \xrightarrow[\sigma']{*} & C[(\lambda x.M)N_n] \\
\Big\downarrow R & & & & \\
C[M[x\backslash N]] & \dashrightarrow & C[M[x\backslash N_1]] & \overset{*}{\dashrightarrow} & C[M[x\backslash N_n]] \\
\Big\downarrow S & & & & \\
C[M^x[N,\ldots N_1\ldots N]] & & & & \\
\sigma \Big\downarrow * & & & & \\
C[M^x[N,\ldots N_n\ldots N]] & & & &
\end{array}
$$

Therefore, if $S\sigma \triangleright \tau$, τ is also in the $i^{\underline{th}}$ instance of N in $M[x\backslash N]$. The reductions $S\sigma$ and $S'\sigma'$ are isomorphic reductions inside M. Thus there is τ' such that $R\tau \triangleright_4^i \tau'$ and $S'\sigma' \triangleright \tau'$. □

FOOTNOTE

1. For example, $F_1F_2\ldots F_n$ will denote the parallel reduction $M \xrightarrow{*} M_n$ above.

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