Sequentiality in Kahn-Macqueen nets and the λ-calculus

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CENTRE DE RECHERCHE COMMUN
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MICROSOFT RESEARCH

Plan

- Kahn-Macqueen networks
- Stability in the λ-calculus
- Stability in Kahn-Macqueen networks
- Revisiting stability in dynamics of the λ-calculus
  - Sequentiality
  - Application to Kahn-Macqueen networks

Edinburgh in 70's

Edinburgh in 70's

Hope Park Square

Scientific visits

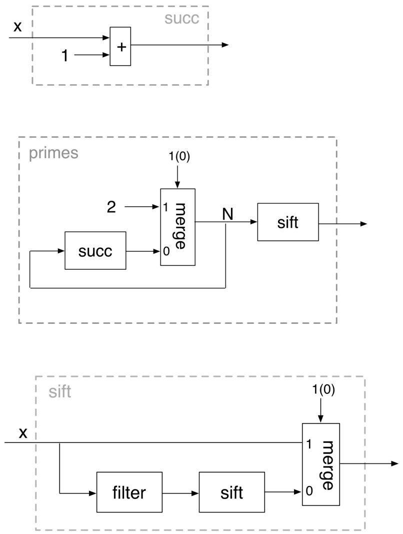
Kahn-Macqueen networks (0/4)

- sieve of Eratosthenes in POP-2 [GK, DBM 77]

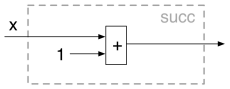
```
Process INTEGERS out Q0;
   Vars N; 1 → N;
   repeat INCREMENT N; PUT(N,Q0) forever
Endprocess;
Process FILTER PRIME in QI out Q0;
   Vars N;
   repeat GET(QI) → N;
        if (N MOD PRIME)≠ ∅ then PUT(N,Q0) close
   forever
Endprocess;
Process SIFT in QI out Q0;
   Vars PRIME; GET(QI) → PRIME;
   PUT (PRIME,Q0); comment emit a discovered prime;
   doco channels Q;
    FILTER(PRIME,QI,Q); SIFT(Q,Q0)
   closeco
Endprocess;
Process OUTPUT in QI; Comment this is a library process;
   repeat PRINT(GET(QI)) forever
Endprocess;
Start doco channels Q1 Q2;
   INTEGERS(Q1); SIFT(Q1,Q2); OUTPUT(Q2);
   closeco;
```

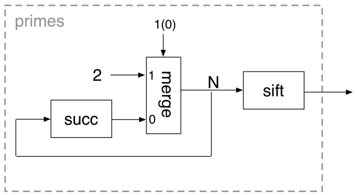
Fig.3. Sieve of Eratosthenes.

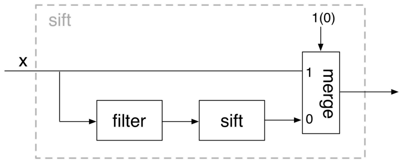
Kahn-Macqueen networks (1/4)



Kahn-Macqueen networks (2/4)








```
succ (x :: xs) := (x+1) :: succ xs

N := 2 :: succ N
primes := sift N


sift (x :: xs) := x :: sift (filter (x :: xs))


filter (x :: xs) := not_mult x xs
not_mult x (y :: ys) :=
  if y mod x = 0 then not_mult x ys
  else x :: (not_mult x ys)
```


Kahn-Macqueen networks (3/4)

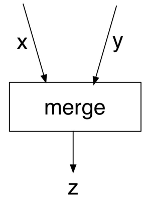
- recursive equations on **flow histories**
- deterministic results (**determinate**)
- problem with «**fair merge**»

$$\text{fmerge}(\text{xs}, \epsilon) = \{\text{xs}\}$$
$$\text{fmerge}(\epsilon, \text{ys}) = \{\text{ys}\}$$
$$\text{fmerge}(x :: \text{xs}, y :: \text{ys}) = \{x :: \text{zs} \mid \text{zs} \in \text{fmerge}(\text{xs}, y :: \text{ys})\} \cup \{y :: \text{zs} \mid \text{zs} \in \text{fmerge}(x :: \text{xs}, \text{ys})\}$$

- equality of traces is **not compositional** [Brock,Ackerman 81]
- powerdomain semantics, process calculi + bisimulations
  [Plotkin 78] [Milner et al 78]

Kahn-Macqueen networks (4/4)

- «merge» **blocks** on its arguments x or y
- since «merge» is **sequential**
- «fair merge» is not sequential like **parallel-or**



$$\text{por}(\text{true}, x) = \text{true}$$
$$\text{por}(x, \text{true}) = \text{true}$$

meaning

$$\text{por}(\text{true}, \bot) = \text{por}(\bot, \text{true}) = \text{true}$$
$$\text{por}(\bot, \bot) = \bot$$

Sequentiality

Scott's semantics - 1st order
strict functions [Cadiou, 71]
alternative def [Vuillemin, 72]

PCF sequential [Plotkin, 75]

stable functions [Berry, 75]

concrete domains [Kahn-Plotkin, 76?]

CDS [Berry-Curien, 79]

⋮

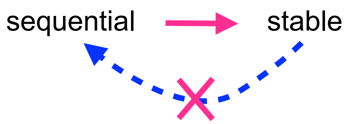
fully abstract models [Abramsky et al, 93]

Stability

- f **stable** function iff $x \uparrow y \;\Rightarrow\; f(x \sqcap y) = f(x) \sqcap f(y)$
- **por** is not stable :

$$\bot = \text{por}(\bot,\bot) \neq \text{por}(\bot,\text{true}) \sqcap \text{por}(\text{true},\bot) = \text{true}$$

- semantics of (strongly) stable functions
- with strange Berry's function



Stability inside calculi

- PCF [Plotkin,75]

$M, N, P ::= x \mid \lambda x.M \mid MN \mid n \mid M \oplus N \mid \texttt{ifz}\ P\ \texttt{then}\ M\ \texttt{else}\ N$

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$
$$\underline{m} \oplus \underline{n} \longrightarrow \underline{m+n}$$
$$\texttt{ifz}\ \underline{0}\ \texttt{then}\ M\ \texttt{else}\ N \longrightarrow M$$
$$\texttt{ifz}\ \underline{n+1}\ \texttt{then}\ M\ \texttt{else}\ N \longrightarrow N$$

- PCF cannot express por.

Stability inside the λ-calculus (1/3)

$$M, N ::= x \mid \lambda x.M \mid MN$$
$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

- Impossible to get:

$$C[\Omega,\Omega] \not\xrightarrow{*} \text{nf}$$
$$C[\Omega,\lambda x.x] \xrightarrow{*} \text{nf}$$
$$C[\lambda x.x,\Omega] \xrightarrow{*} \text{nf}$$

Lemma «has a nf» is a stable function.

Stability inside the λ-calculus (2/3)

$$M, N ::= x \mid \lambda x.M \mid MN$$
$$(\lambda x.M)N \longrightarrow M\{x := N\}$$

- Impossible to get:

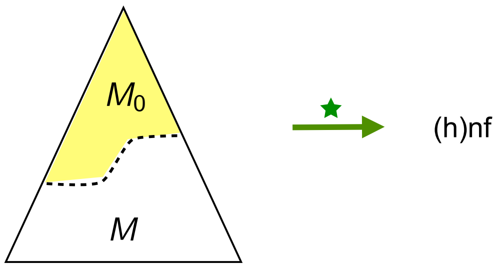
$$C[\Omega,\Omega] \not\xrightarrow{*} \text{hnf}$$
$$C[\Omega,H'] \xrightarrow{*} \text{hnf}$$
$$C[H,\Omega] \xrightarrow{*} \text{hnf} \qquad (H,\ H' \text{ with hnf})$$

Lemma «has a hnf» is a stable function.

Lemma «Bohm tree» is a stable function.

Stability inside the λ-calculus (3/3)

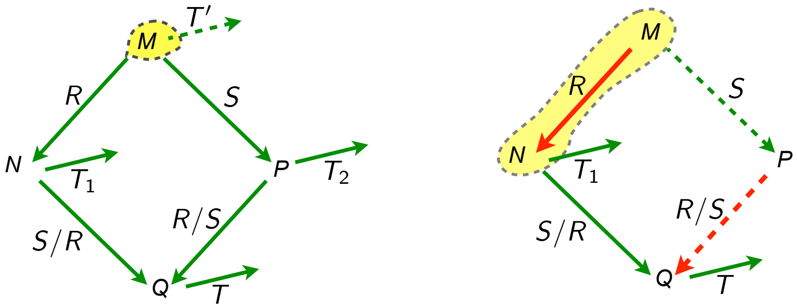
Lemma Let $M \xrightarrow{\star}$ (h)nf, then there is a unique minimum prefix M_0 of M such that $M_0 \xrightarrow{\star}$ (h)nf.



Stability inside redexes (1/2)

Lemma [stability of redex creation] When $R \neq S$,

$T \in T_1/(S/R)$ and $T \in T_2/(R/S)$ implies $T \in T'/(R \sqcup S)$



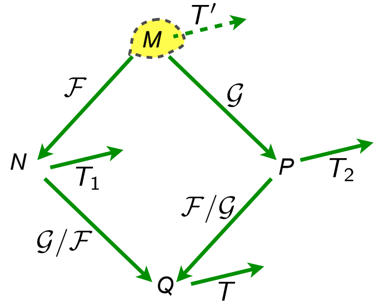
Corollary When $R \neq S$,

If $T \in T_1/(S/R)$ and R creates T_1, then $\exists R' \in R/S$, R' creates T.

Stability inside redexes (2/2)

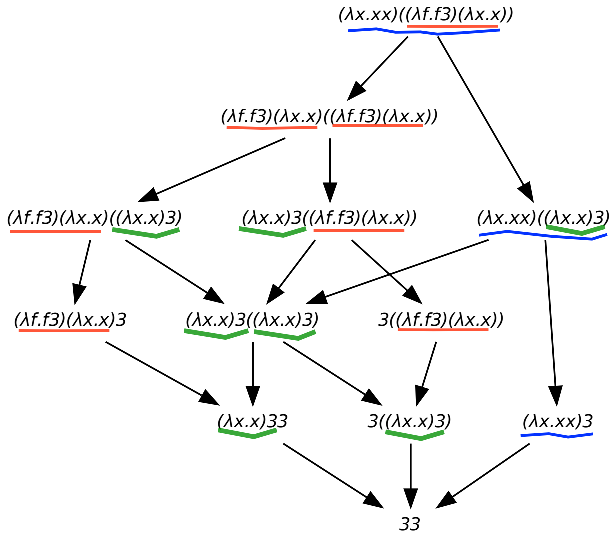
Lemma [stability of redex creation] When $\mathcal{F} \cap \mathcal{G} = \emptyset$,

$T \in T_1/(\mathcal{G}/\mathcal{F})$ and $T \in T_2/(\mathcal{F}/\mathcal{G})$ implies $T \in T'/(\mathcal{F} \sqcup \mathcal{G})$



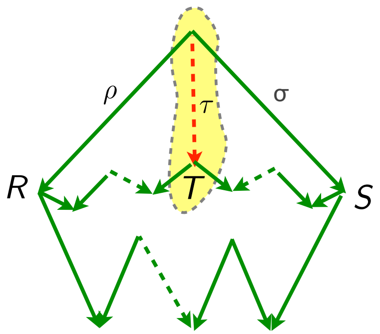
Corollary When $\mathcal{F} \cap \mathcal{G} = \emptyset$, if $\mathcal{F}$ creates T, then $\mathcal{G}/\mathcal{F}$ creates $T/(\mathcal{G}/\mathcal{F})$.

Redex families



- 3 redex families: **red**, **blue**, **green**.

Redexes and their unique origin



Proposition

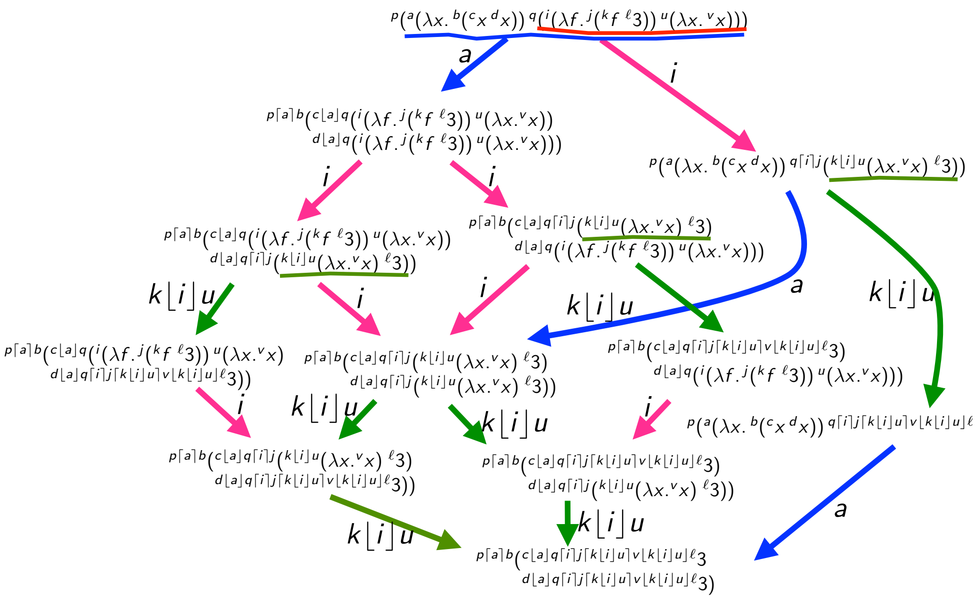
There is a unique $\langle \tau, T \rangle$ with τ standard reduction of minimum length in each redex family.

Stability in Kahn-Macqueen nets

- Equations on history flows are left-linear orthogonal TRS
- Stability for prefixes `[Huet, JJL, 81; Klop 90]`
- Stability inside their redexes `[Maranget 91]`

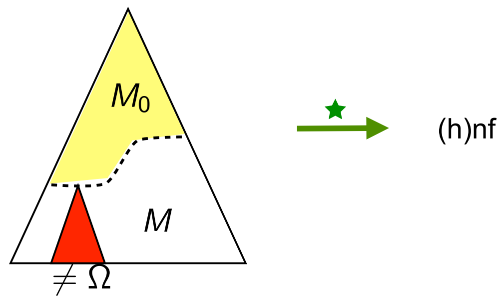
Redex families

3 families and their names: a i $k\lfloor i \rfloor u$



Sequentiality (1/2)

Lemma Let $M_0 \not\xrightarrow{\star}$ (h)nf, then there is an Ω occurrence such that you cannot get a (h)nf without strictly increasing it.



Sequentiality (2/2)

- «Bohm-tree» is a sequential function [Berry, JJL, 78]

  $C[\Omega, \Omega] \not\xrightarrow{*}$ nf

  $C[M, N] \xrightarrow{*}$ nf for some M and N

  one of the Ω's is such that $C[\Omega, N] \not\xrightarrow{*}$ nf for all N

- Theory of strongly sequential TRS
  [Huet, JJL, 81, Klop 90]

- Call by need calculations for Kahn-Macqueen nets

Todo-list

- From strongly sequential TRS to Kahn-Macqueen networks
- Theory of sequentiality for redexes
- Need to work with subcontexts ?

Enjoy retirement Dave!

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